INTERSECTION TYPES FOR COMBINATORY LOGIC

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Dedicated to J. W. de Bakker in honour of his 25 years of work in semantics.

ABSTRACT.

Two different translations of the usual formulation of intersection types for λ-calculus into combinatory logic are proposed; in the first one the rule $(\leq)$ is unchanged, while in the second one the rule $(\leq)$ is replaced by three new rules and five axiom-schemes, which seem to be simpler than rule $(\leq)$ itself.

INTRODUCTION.

Intersection types were introduced as a generalization of the type discipline of Church and Curry, mainly with the aim of describing the functional behaviour of all solvable λ-terms. The usual $\rightarrow$-based type-language for λ-calculus was extended by adding a constant ω as a universal type and a new connective $\wedge$ for the intersection of two types. With suitable axioms and rules to assign types to λ-terms, this gave a system in which (i) the set of types given to a λ-term does not change under β-conversion, and (ii) the sets of normalizing and solvable λ-terms can be characterized very neatly by the types of their members. (CDV[1981] gives an introduction and motivation of $\wedge$ and ω, and BCD[1983] gives a summary of all the most basic syntactic properties of the system.)

Moreover, in the new type-language we can build λ-models (filter models) in which the interpretation of a λ-term coincides with the set af all types that can be assigned to it. Filter models turn out to be a very rich class containing in particular each inverse-limit space, and have been widely used to study properties of D_∞-λ-models; see BCD[1983], CDHL[1983] and CDZ[1987].

More recently, intersection types have been introduced in the programming language Forsythe, which is a descendent of Algol 60, to simplify the structure of types; see R[1988].

Systems of combinators are designed to perform the same tasks as systems of λ-calculus, but without using bound variables. Curry's type discipline turns out to be significantly simpler in combinatory logic than in λ-calculus. (For an introduction see HS[1986] Chapter 14.)

We propose here two different formulations of intersection types for combinatory logic. They are both essentially just translations of the λ-calculus system presented in BCD[1983], and have all the properties one would expect. However, there is at least one extra complication in combinatory logic. In the case of λ-calculus, the type-assignment rule $(\leq)$ is well known to be replaceable by the simpler rule (η) (§1 below). But in combinatory logic some more care must be taken in choosing a rule to replace rule $(\leq)$, and we do not know whether the second system we present below is the simplest possible (see §4).

For background λ-calculus, combinatory logic and type-theory, HS[1986] will be used as a basic reference.

1. INTERSECTION TYPES FOR λ-CALCULUS.

We introduce the intersection type-assigment system following BCD[1983], H[1982] and H[1988].

1.1 DEFINITION. (i) The set T of *intersection types* is inductively defined by:

$\phi_0, \phi_1, \ldots \in T$ (type-variables)

$\omega \in T$ (type-constant)

$\sigma,\tau \in T \;\Rightarrow\; (\sigma\to\tau) \in T,\; (\sigma\wedge\tau) \in T.$

(ii) A *(type-assignment) statement* is of the form $M{:}\sigma$ with $\sigma \in T$ and M a λ-term, called its *subject*. A *basis* B is a set of statements with only distinct variables as subjects. If x does not occur in B, then "B, $x{:}\sigma$" denotes $B\cup\{x{:}\sigma\}$.

On intersection types we define a pre-order relation which formalizes the subset relation and will be used in a type-assignment rule.

1.2 DEFINITION. The *$\leq$ relation* on intersection types is inductively defined by:

$$\begin{array}{ll} \tau\leq\tau, & \tau\leq\tau\wedge\tau, \\ \tau\leq\omega, & \sigma\wedge\tau\leq\sigma,\quad \sigma\wedge\tau\leq\tau, \\ \omega\leq\omega\to\omega, & (\sigma\to\rho)\wedge(\sigma\to\tau)\leq\sigma\to(\rho\wedge\tau), \end{array}$$

$$\begin{array}{lll} \sigma\leq\rho\leq\tau & \Rightarrow & \sigma\leq\tau, \\ \sigma\leq\sigma',\ \tau\leq\tau' & \Rightarrow & \sigma\wedge\tau\leq\sigma'\wedge\tau', \\ \sigma\leq\sigma',\ \tau\leq\tau' & \Rightarrow & \sigma'\to\tau\leq\sigma\to\tau'. \end{array}$$

1.3 DEFINITION. (i) $TA_\lambda(\wedge,\omega,\leq)$ is the type assignment system defined by the following natural-deduction rules and axioms.

Axioms (ω): $\quad M{:}\omega \quad$ (one axiom for each λ-term M).

Rules:

$$(\to I)\ \frac{\begin{matrix}[x{:}\sigma]\\ \vdots \\ M{:}\tau\end{matrix}}{\lambda x.M{:}\sigma\to\tau}\ (*) \qquad\qquad (\to E)\ \frac{M{:}\sigma\to\tau \qquad N{:}\sigma}{MN{:}\tau}$$

$$(\wedge I)\ \frac{M{:}\sigma \qquad M{:}\tau}{M{:}\sigma\wedge\tau} \qquad\qquad (\wedge E)\ \frac{M{:}\sigma\wedge\tau}{M{:}\sigma} \qquad \frac{M{:}\sigma\wedge\tau}{M{:}\tau}$$

$$(\leq)\ \frac{M{:}\sigma \qquad \sigma\leq\tau}{M{:}\tau}$$

(*) if x is not free in assumptions above $M{:}\tau$, other than $x{:}\sigma$.

(ii) We write $B \vdash_\lambda M{:}\sigma$ if $M{:}\sigma$ is derivable from the basis B in this system.

The main syntactic property of this type system is the following theorem of invariance under β-equality and η-reduction. (For a proof see CDV[1981] Lemma 1 and Theorem 1, or H[1982] §5.)

1.5 THEOREM. (i) $TA_\lambda(\wedge,\omega,\leq)$ *is invariant under* β*-equality; that is, if* $M =_\beta N$ *and* $B \vdash_\lambda M{:}\sigma$, *then* $B \vdash_\lambda N{:}\sigma$.

(ii) $TA_\lambda(\wedge,\omega,\leq)$ *is invariant under* η*-reduction; that is, if* $z \notin FV(M)$ *and* z *does not occur in* B, *and* $B, z{:}\sigma \vdash_\lambda Mz{:}\tau$, *then* $B \vdash_\lambda M{:}(\sigma\to\tau)$.

The invariance under η-reduction allows a replacement of rule ($\leq$) which preserves type assignment, as follows.

1.6 DEFINITION. (i) Let $TA_\lambda(\wedge,\omega,\eta)$ be the type-assigment system obtained from $TA_\lambda(\wedge,\omega,\leq)$ by replacing rule $(\leq)$ by

$$(\eta)\quad \frac{(\lambda x.Mx):\sigma}{M:\sigma\ .}\ \text{(if x is not free in M)}$$

(ii) Let $B\vdash_{\lambda\eta} M:\sigma$ denote derivability in the resulting system.

1.7 THEOREM. $TA_\lambda(\wedge,\omega,\leq)$ *and* $TA_\lambda(\wedge,\omega,\eta)$ *are equivalent; that is,*

$$B\vdash_\lambda M:\sigma \quad\Leftrightarrow\quad B\vdash_{\lambda\eta} M:\sigma.$$

This equivalence can be proved directly fairly easily, or by using BCD[1983] (in particular Lemma 4.2, Remark 2.10, and the remark just before 4.3).

2. CORRESPONDENCE BETWEEN λ AND CL.

The reader is assumed to know at least the basic definitions of combinatory logic (see Chapter 2 of HS[1986]). The atomic combinators are assumed here to be **S**, **K**, **I**.

2.1 DEFINITION (*Abstraction in Combinatory Logic*).

(i) A *functional (fnl)* term is any of $\mathbf{S}$, $\mathbf{S}X$, $\mathbf{S}XY$, $\mathbf{K}$, $\mathbf{K}X$, $\mathbf{I}$ (for any X,Y).

(ii) We present four alternative definitions for $\lambda^*x.X$. (The second one has been discussed in HS[1986] §§9.34-35, and the other three are common in the literature. Note that the definition of λ^β uses λ^η.)

λ^η: (a) $\lambda^\eta x.Y \equiv \mathbf{K}Y$ if $x\notin FV(Y)$,
(b) $\lambda^\eta x.x \equiv \mathbf{I}$,
(c) $\lambda^\eta x.Ux \equiv U$ if $x\notin FV(U)$,
(f) $\lambda^\eta x.UV \equiv \mathbf{S}(\lambda^\eta x.U)(\lambda^\eta x.V)$ if (a)-(c) do not apply.

λ^β: (a), (b) as above,
(c_β) $\lambda^\beta x.Ux \equiv U$ if $x\notin FV(U)$ and U is fnl,
(f_β) $\lambda^\beta x.UV \equiv \mathbf{S}(\lambda^\eta x.U)(\lambda^\eta x.V)$ if (a)-(c_β) do not apply.

λ^{abf}: (a), (b) as above, and (f) used when (a) and (b) do not apply.

λ^{fab} : (f) $\lambda^{fab}x.UV \equiv \mathbf{S}(\lambda^{fab}x.U)(\lambda^{fab}x.V)$,
(a) $\lambda^{fab}x.y \equiv \mathbf{K}y$ if y is an atom distinct from x,
(b) $\lambda^{fab}x.x \equiv \mathbf{I}$.

2.2 DEFINITION (*H-transformations*). Each abstraction determines an H-mapping from λ-calculus to combinatory logic: $(\lambda x.M)_H \equiv \lambda^* x.(M_H)$. (Details are in HS[1986] Chapter 9.) We call these mappings H_β, H_η, H_{abf}, H_{fab}.

Let X_λ denote the λ-term associated in the standard way with the CL-term X, and let $=_{c\beta}$ denote combinatory β-equality (i.e. $X =_{c\beta} Y \Leftrightarrow X_\lambda =_\beta Y_\lambda$).

2.3 LEMMA. (i) *For all CL-terms* X:

$X_{\lambda H\eta} \equiv X$, *in particular* $\mathbf{S}_{\lambda H\eta} \equiv \mathbf{S}$;

$X_{\lambda H\beta} \equiv X$, *in particular* $\mathbf{S}_{\lambda H\beta} \equiv \mathbf{S}$;

$X_{\lambda Habf} =_{c\beta} X$ *and* $\mathbf{S}_{\lambda Habf} \not\equiv \mathbf{S}$;

$X_{\lambda Hfab} =_{c\beta} X$ *and* $\mathbf{S}_{\lambda Hfab} \not\equiv \mathbf{S}$.

(ii) *For all λ-terms* M *and for* H_β *or* H_{abf} *or* H_{fab} : $M_{H\lambda} =_\beta M$.

The proof for H_{abf} is in HS[1986] §§9.20-28, and the others are similar; see HS[1986] §9.35 for hints on the proof for H_β.

3. INTERSECTION TYPES FOR CL-TERMS.

We introduce now an assignment of intersection types to CL-terms which can be viewed as a translation of $TA_\lambda(\wedge,\omega,\leq)$ into combinatory logic. Its relation to $TA_\lambda(\wedge,\omega,\leq)$ will be precisely stated in Theorem 3.3.

In this section, *type-assignment statements* have form $X{:}\sigma$ where X is a CL-term. *Bases* are sets $\{x_1{:}\sigma_1, x_2{:}\sigma_2, \ldots\}$ with $x_1, x_2, \ldots$ distinct, as usual.

3.1 DEFINITION. (i) $TA_{CL\beta}(\wedge,\omega,\leq)$ is the system whose rules are $(\to E)$, $(\wedge I)$, $(\wedge E)$, $(\leq)$, and whose axiom-schemes are (ω) and

$(\to\mathbf{I})$ $\mathbf{I}: \sigma\to\sigma$,

$(\to\mathbf{K})$ $\mathbf{K}: \sigma\to\tau\to\sigma$,

$(\to\mathbf{S})$ $\mathbf{S}: (\sigma\to\tau\to\rho)\to(\sigma\to\tau)\to\sigma\to\rho$.

(ii) We write $B \vdash_{CL} X{:}\sigma$ if $X{:}\sigma$ is derivable from the basis B in this system.

3.2 LEMMA. (i) $B, x{:}\sigma \vdash_{CL} x{:}\tau \;\Rightarrow\; \sigma \leq \tau$.

(ii) *Let* λ^* *be any of* λ^η, λ^β, λ^{abf}, λ^{fab}. *Then*

$$B, x{:}\sigma \vdash_{CL} Y{:}\tau \;\Rightarrow\; B \vdash_{CL} (\lambda^*x.Y){:}\sigma \to \tau.$$

Proof. (i) By an easy induction on deductions.

(ii) Induction on the deduction of $Y{:}\tau$. We will prove the result for all four λ^*'s at once, and will use the induction hypothesis for λ^η in proving the induction step for λ^β.

Case 1: $Y{:}\tau$ is $x{:}\sigma$. $\therefore$ $Y \equiv x$, $\therefore$ $\lambda^*x.Y \equiv \mathbf{I}$. But $\mathbf{I}{:}\sigma \to \sigma$ is an axiom.

Case 2: $Y{:}\tau$ is either in B, or is an $\mathbf{S}$, $\mathbf{K}$ or $\mathbf{I}$ axiom, or an ω-axiom with Y an atom $\not\equiv x$. $\therefore$ Y is an atom and $x \notin FV(Y)$, so $\lambda^*x.Y \equiv \mathbf{K}Y$. Hence, by the axiom $\mathbf{K}{:}\tau \to \sigma \to \tau$ and rule $(\to E)$, $B \vdash_{CL} \mathbf{K}Y{:}\sigma \to \tau$.

Case 3: $Y{:}\tau$ is an ω-axiom. $\therefore$ $\tau \equiv \omega$. Now $(\lambda^*x.Y){:}\omega$ is an ω-axiom. And, since $\sigma \leq \omega$, we have $\omega \leq \omega \to \omega \leq \sigma \to \omega$. Hence $(\lambda^*x.Y){:}\sigma \to \omega$ by rule $(\leq)$.

Case 4: The last step in the deduction of $Y{:}\tau$ is $(\leq)$ or $(\wedge E)$:

$$\begin{array}{c} B \quad x{:}\sigma \\ \vdots \\ \dfrac{Y{:}\rho}{Y{:}\tau} \;(\rho \leq \tau) \end{array}$$

Then $(\sigma \to \rho) \leq (\sigma \to \tau)$, so we use the induction hypothesis and rule $(\leq)$.

Case 5: Rule $(\wedge I)$:

$$\frac{Y{:}\tau_1 \qquad Y{:}\tau_2}{Y{:}\,(\tau_1 \wedge \tau_2)} \quad (\tau \equiv \tau_1 \wedge \tau_2)$$

By induction hypothesis, $B \vdash_{CL} (\lambda^*x.Y){:}\sigma \to \tau_i$ for $i = 1,2$. But $(\sigma \to \tau_1) \wedge (\sigma \to \tau_2) \leq \sigma \to (\tau_1 \wedge \tau_2)$, so rules $(\wedge I)$ and $(\leq)$ give the result.

Case 6: Rule $(\to E)$: Say $Y \equiv UV$, and we have:

$$\dfrac{\begin{array}{c} B \quad x{:}\sigma \\ \vdots \\ U{:}\rho \to \tau \end{array} \qquad \begin{array}{c} B \quad x{:}\sigma \\ \vdots \\ V{:}\rho \end{array}}{UV{:}\tau}$$

Subcase 6a: $x \notin FV(UV)$ and $\lambda^*x.(UV) \equiv \mathbf{K}(UV)$. Since $x \notin FV(UV)$, x cannot occur in the given deduction. Hence $B \vdash_{CL} UV{:}\tau$. So by the axiom $\mathbf{K}{:}\tau \to \sigma \to \tau$ and rule $(\to E)$, $B \vdash_{CL} \mathbf{K}(UV){:}\sigma \to \tau$.

Subcase 6c: $V \equiv x$, $x \notin FV(U)$, and $\lambda^*x.(UV) \equiv U$. Since $B, x{:}\sigma \vdash x{:}\rho$, we have $\sigma \leq \rho$ by (i). $\therefore (\rho \to \tau) \leq (\sigma \to \tau)$. But $B \vdash_{CL} U{:}(\rho \to \tau)$ since $x \notin FV(U)$; hence by $(\leq)$, $B \vdash_{CL} U{:}(\sigma \to \tau)$.

Subcase 6f: $\lambda^*x.(UV) \equiv \mathbf{S}(\lambda^{*\prime}x.U)(\lambda^{*\prime}x.V)$ (where $\lambda^{*\prime}$ is λ^* if λ^* is λ^η or λ^{abf} or λ^{fab}, but $\lambda^{*\prime}$ is λ^η if λ^* is λ^β). By induction hypothesis for $\lambda^{*\prime}$, we have $B \vdash_{CL} (\lambda^{*\prime}x.U){:}\sigma \to \rho \to \tau$, $B \vdash_{CL} (\lambda^{*\prime}x.V){:}\sigma \to \rho$. Hence the result, by an $\mathbf{S}$-axiom and $(\to E)$. □

3.3 THEOREM. (i) $B \vdash_{CL} X{:}\tau \iff B \vdash_\lambda X_\lambda{:}\tau$.

(ii) $B \vdash_\lambda M{:}\tau \Rightarrow B \vdash_{CL} M_H{:}\tau$ *for* $H_\eta, H_\beta, H_{abf}, H_{fab}$.

(iii) *For* H_β, H_{abf}, H_{fab}, *we also have the converse of (ii).*

Proof. We prove all parts together. (i) "$\Rightarrow$" is trivial.

(ii): Induction on $\vdash_\lambda$. The only difficult case is rule $(\to I)$, which comes by Lemma 3.2.

(iii): Let H be any of H_β, H_{abf}, H_{fab}, and let $B \vdash_{CL} M_H{:}\tau$. $\therefore$ by (i)"$\Rightarrow$", $B \vdash_\lambda M_{H\lambda}{:}\tau$. But $M_{H\lambda} =_\beta M$ by Lemma 2.3(ii). $\therefore$ by Theorem 1.5(i), $B \vdash_\lambda M{:}\tau$.

(i)"$\Leftarrow$": Let $B \vdash_\lambda X_\lambda{:}\tau$. Then $B \vdash_{CL} X_{\lambda H_\beta}{:}\tau$ by (ii). $\therefore$ $B \vdash_{CL} X{:}\tau$ because $X_{\lambda H_\beta} \equiv X$ by Lemma 2.3(i). □

Note that Theorem 3.3(iii) does not hold for H_η. A counter-example is $M \equiv \lambda xy.xy$; we have $M_{H_\eta} \equiv \mathbf{I}$ which has type $\phi \to \phi$ in the CL-system (ϕ being a type-variable), but it can be shown that M does not have this type in the λ-system.

The following theorem shows that $TA_{CL\beta}(\wedge,\omega,\leq)$ is invariant under β-equality and η-reduction.

3.4 THEOREM. (i) *If* $B \vdash_{CL} X{:}\tau$ *and* $Y =_{c\beta} X$, *then* $B \vdash_{CL} Y{:}\tau$.

(ii) *If* $B, z{:}\sigma \vdash_{CL} Yz{:}\tau$ *and* $z \notin FV(Y)$ *and* z *is not in* B, *then* $B \vdash_{CL} Y{:}(\sigma \to \tau)$.

Proof. (i): By 3.3(i), (iii) and 1.5(i).

(ii) Induction on the deduction of $Yz{:}\tau$, as follows.

Axioms: $Yz{:}\tau$ cannot be an **S**, **K**, **I**-axiom. The only possibility is an ω-axiom, with $\tau \equiv \omega$. But $\omega \le \omega\to\omega \le \sigma\to\omega$ (since $\sigma \le \omega$), so we have

$$\dfrac{\begin{array}{c}(\omega)\text{-ax}\\ Y{:}\omega\end{array}}{Y{:}(\sigma\to\omega)}\ (\omega \le \sigma\to\omega) \quad .$$

Rule $(\to E)$: Say we have, for some ρ,

$$\dfrac{Y{:}\rho\to\tau \qquad z{:}\rho}{Yz{:}\tau.}$$

But $z{:}\rho$ is deduced from B, $z{:}\sigma$ and z does not occur in B. Hence $\sigma \le \rho$ by 3.2(i). $\therefore$ $(\rho\to\tau) \le (\sigma\to\tau)$, so by $Y{:}(\rho\to\tau)$ and rule $(\le)$, $B \vdash_{CL} Y{:}(\sigma\to\tau)$.

Rule $(\le)$ or $(\wedge E)$: Say we have

$$\dfrac{Yz{:}\rho}{Yz{:}\tau}\ (\rho \le \tau) \quad .$$

By induction hypothesis, $B \vdash_{CL} Y{:}(\sigma\to\rho)$. Hence, by $(\le)$, $B \vdash_{CL} Y{:}(\sigma\to\tau)$.

Rule $(\wedge I)$: Say $\tau \equiv (\tau_1\wedge\tau_2)$ and we have

$$\dfrac{Yz{:}\tau_1 \qquad Yz{:}\tau_2}{Yz{:}\,(\tau_1\wedge\tau_2)} \quad .$$

By induction hypothesis, $B \vdash_{CL} Y{:}(\sigma\to\tau_i)$, $i = 1,2$. $\therefore$ by $(\wedge I)$ and $(\le)$, since $(\sigma\to\tau_1)\wedge(\sigma\to\tau_2) \le \sigma\to(\tau_1\wedge\tau_2)$, we have $B \vdash_{CL} Y{:}\sigma\to(\tau_1\wedge\tau_2)$. □

3.5 NOTE. Following H[1982], let us define the set NTS of *Normal Types* to be the set of all types σ such that: either $\sigma \equiv \omega$ or $\sigma \equiv \sigma_1\wedge\ldots\wedge\sigma_n$ with some bracketing and with each σ_i having the form $\sigma_{i,1}\to\ldots\to\sigma_{i,m(i)}\to\phi_i$. Normal types corresponded closely to the types in CDV[1981], which were slightly more restricted than those in BCD[1983] and later papers, including this one. In H[1982] it was proved that the restriction was trivial, in the sense that every deduction $B \vdash_\lambda M{:}\tau$ could be paralleled by a deduction $B^* \vdash_\lambda M{:}\tau^*$ containing only normal types, where the map $*$: $T\to$NTS applied to a type gave its "normal form". But in CL the restriction seems not to be so trivial. For example, in CL there is a problem with the axiom $\mathbf{I}:(\sigma\wedge\tau)\to(\sigma\wedge\tau)$. The type in this is not normal, and the nearest normal type to it is $((\sigma\wedge\tau)\to\sigma)\wedge((\sigma\wedge\tau)\to\tau)$. So if types were restricted to being normal, quite a complicated form of the axiom scheme for $\mathbf{I}$ would be needed to give a reasonable equivalence to the λ-system. Similarly for $\mathbf{S}$ and $\mathbf{K}$.

4. REPLACING RULE ($\leq$).

In this section we propose an alternative formulation of intersection type-assignment to CL-terms in which rule ($\leq$) has been replaced by something simpler. Let $\mathbf{B} \equiv \mathbf{S}(\mathbf{KS})\mathbf{K}$ and $\mathbf{B}' \equiv \mathbf{SB}(\mathbf{KI})$.

4.1 DEFINITION. (i) $\mathrm{TA}_{\mathrm{CL}\beta}(\wedge,\omega,\eta)$ is the system for CL-terms whose axiom-schemes are (ω), $(\to\mathbf{I})$, $(\to\mathbf{K})$, $(\to\mathbf{S})$ and

$(\mathbf{I}_1)$	$\mathbf{I}:\sigma\to\omega$	
$(\mathbf{I}_2)$	$\mathbf{I}:\omega\to(\omega\to\omega)$	
$(\mathbf{I}_3)$	$\mathbf{I}:(\sigma_1\wedge\sigma_2)\to\sigma_i$	$(i = 1,2)$
$(\mathbf{I}_4)$	$\mathbf{I}:((\sigma\to\tau)\wedge(\sigma\to\rho))\to(\sigma\to(\tau\wedge\rho))$	

and whose rules are $(\to\mathrm{E})$, $(\wedge\mathrm{I})$, $(\wedge\mathrm{E})$ and

$$(\mathbf{I}_5)\ \frac{\mathbf{I}X:\sigma}{X:\sigma} \qquad (\eta_1)\ \frac{\mathbf{BI}:\sigma}{\mathbf{I}:\sigma} \qquad (\eta_2)\ \frac{\mathbf{B}'\mathbf{I}:\sigma}{\mathbf{I}:\sigma}$$

(ii) We write $B \vdash_{\mathrm{CL}\eta} X:\sigma$ if $X:\sigma$ is derivable from the basis B in this system.

We shall prove that $\mathrm{TA}_{\mathrm{CL}\beta}(\wedge,\omega,\leq)$ and $\mathrm{TA}_{\mathrm{CL}\beta}(\wedge,\omega,\eta)$ are equivalent.

4.2 LEMMA. *If* $\sigma \leq \sigma'$, *then* $\vdash_{\mathrm{CL}\eta} \mathbf{I}:\sigma\to\sigma'$.

<u>Proof</u>. Induction on the proof of $\sigma\leq\sigma'$. We consider only the non-trivial cases.
<u>Axiom</u> $\sigma \leq \sigma\wedge\sigma$.

$$
\dfrac{\overset{(\mathbf{I}_4)\text{-ax}}{\mathbf{I}:((\sigma\to\sigma)\wedge(\sigma\to\sigma))\to\sigma\to(\sigma\wedge\sigma)} \qquad \dfrac{\overset{(\mathbf{I})\text{-ax}}{\mathbf{I}:\sigma\to\sigma} \qquad \overset{(\mathbf{I})\text{-ax}}{\mathbf{I}:\sigma\to\sigma}}{\mathbf{I}:(\sigma\to\sigma)\wedge(\sigma\to\sigma)}\,(\wedge I)}{\dfrac{\mathbf{II}:\sigma\to(\sigma\wedge\sigma)}{\mathbf{I}:\sigma\to(\sigma\wedge\sigma)}\,(\mathbf{I}_5)}\,(\to E)
$$

Transitivity: Suppose $\mathbf{I}:\sigma\to\tau$ and $\mathbf{I}:\tau\to\rho$. Deduce $\mathbf{I}:\sigma\to\rho$ thus:

$$
\dfrac{\dfrac{\dfrac{\mathbf{B}:(\tau\to\rho)\to(\sigma\to\tau)\to\sigma\to\rho \qquad \mathbf{I}:\tau\to\rho}{\mathbf{BI}:(\sigma\to\tau)\to\sigma\to\rho}\,(\to E)}{\mathbf{I}:(\sigma\to\tau)\to\sigma\to\rho}\,(\eta_1) \qquad \mathbf{I}:\sigma\to\tau}{\dfrac{\mathbf{II}:\sigma\to\rho}{\mathbf{I}:\sigma\to\rho}\,(\mathbf{I}_5)}\,(\to E)\ .
$$

Replacement in $\wedge$. Assume $\mathbf{I}:\sigma\to\sigma'$ and $\mathbf{I}:\tau\to\tau'$. Deduce $\mathbf{I}:(\sigma\wedge\tau)\to(\sigma'\wedge\tau')$ thus:

$$
\dfrac{\overset{(\mathbf{I}_4)\text{-ax}}{\mathbf{I}:((\sigma\wedge\tau)\to\sigma')\wedge((\sigma\wedge\tau)\to\tau')\to(\sigma\wedge\tau)\to(\sigma'\wedge\tau')} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathbf{B}:(\sigma\to\sigma')\to((\sigma\wedge\tau)\to\sigma)\to(\sigma\wedge\tau)\to\sigma' \qquad \mathbf{I}:\sigma\to\sigma'}{\mathbf{BI}:((\sigma\wedge\tau)\to\sigma)\to(\sigma\wedge\tau)\to\sigma'}\,(\to E)}{\mathbf{I}:((\sigma\wedge\tau)\to\sigma)\to(\sigma\wedge\tau)\to\sigma'}\,(\eta_1) \qquad \overset{(\mathbf{I}_3)\text{-ax}}{\mathbf{I}:(\sigma\wedge\tau)\to\sigma}}{\mathbf{II}:(\sigma\wedge\tau)\to\sigma'}\,(\to E)}{\mathbf{I}:(\sigma\wedge\tau)\to\sigma'}\,(\mathbf{I}_5) \qquad \dfrac{\overset{\vdots}{\text{SIMILAR}}}{\mathbf{I}:(\sigma\wedge\tau)\to\tau'}}{\mathbf{I}:((\sigma\wedge\tau)\to\sigma')\wedge((\sigma\wedge\tau)\to\tau')}\,(\wedge I)}{\dfrac{\mathbf{II}:(\sigma\wedge\tau)\to(\sigma'\wedge\tau')}{\mathbf{I}:(\sigma\wedge\tau)\to(\sigma'\wedge\tau')}\,(\mathbf{I}_5)}\,(\to E)\ .
$$

Replacement in $\to$. Assume $\mathbf{I}:\sigma\to\sigma'$ and $\mathbf{I}:\tau\to\tau'$. Deduce $\mathbf{I}:(\sigma'\to\tau)\to(\sigma\to\tau')$ as follows. In this deduction, let $\xi\equiv(\sigma\to\tau)$, $\eta\equiv(\sigma\to\tau')$, and $\zeta\equiv(\sigma'\to\tau)$.

$$
\dfrac{\dfrac{\dfrac{\mathbf{B}:(\xi\to\eta)\to(\zeta\to\xi)\to\zeta\to\eta \qquad \dfrac{\dfrac{\mathbf{B}:(\tau\to\tau')\to(\sigma\to\tau)\to\sigma\to\tau' \qquad \mathbf{I}:\tau\to\tau'}{\mathbf{BI}:(\sigma\to\tau)\to\sigma\to\tau'}\,(\to E)}{\mathbf{I}:\xi\to\eta}\,(\eta_1)}{\mathbf{BI}:(\zeta\to\xi)\to\zeta\to\eta}\,(\to E)}{\mathbf{I}:(\zeta\to\xi)\to\zeta\to\eta}\,(\eta_1) \qquad \dfrac{\dfrac{\mathbf{B'}:(\sigma\to\sigma')\to(\sigma'\to\tau)\to\sigma\to\tau \qquad \mathbf{I}:\sigma\to\sigma'}{\mathbf{B'I}:(\sigma'\to\tau)\to\sigma\to\tau}\,(\to E)}{\mathbf{I}:\zeta\to\xi}\,(\eta_2)}{\dfrac{\mathbf{II}:\zeta\to\eta}{\mathbf{I}:\zeta\to\eta}\,(\mathbf{I}_5)}\,(\to E)\ . \qquad \square
$$

4.3 THEOREM. $B \vdash_{CL} X{:}\sigma \iff B \vdash_{CL\eta} X{:}\sigma$.

Proof. "$\Rightarrow$": The only thing to show is that $(\leq)$ is an admissible rule in $TA_{CL\beta}(\wedge,\omega,\eta)$; that is, to show that if $B \vdash_{CL\eta} X{:}\sigma$ and $\sigma \leq \tau$, then $B \vdash_{CL\eta} X{:}\tau$. By Lemma 4.2, $\vdash_{CL\eta} \mathbf{I}{:}\sigma\to\tau$. Then we can deduce

$$\dfrac{\dfrac{\mathbf{I}{:}\sigma\to\tau \qquad X{:}\sigma}{\mathbf{I}X{:}\tau}\,(\to E)}{X{:}\tau}\,(\mathbf{I}_5)\ .$$

"$\Leftarrow$": Immediate from 3.4(ii). $\square$

4.4 NOTE. Rule $(\leq)$ can also be replaced by a strengthened $\mathbf{I}$-axiom-scheme saying $\mathbf{I}{:}\sigma\to\tau$ $(\sigma \leq \tau)$, and an $\mathbf{I}$-rule:

$$\dfrac{\mathbf{I}X{:}\sigma}{X{:}\sigma}\ .$$

Using this axiom-scheme and rule, we get $X{:}\sigma \vdash X{:}\tau$ when $\sigma \leq \tau$, as follows:

$$\dfrac{\dfrac{\mathbf{I}{:}\sigma\to\tau \qquad X{:}\sigma}{\mathbf{I}X{:}\tau}\,(\to E)}{X{:}\tau}\ .$$

Conversely, the axiom and $\mathbf{I}$-rule are easily proved admissible in $TA_{CL\beta}(\wedge,\omega,\leq)$.

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